Comparing different formulations of non-linear cosmological perturbation theory

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Abstract. We compare and contrast two different metric based formulations of non-linear cosmological perturbation theory: the MW2009 approach in [K. A. Malik and D. Wands, Phys. Rept. **475** (2009), 1.] following Bardeen and the recent approach of the paper KN2010 [K. Nakamura, Advances in Astronomy **2010** (2010), 576273]. We present each formulation separately. In the MW2009 approach, one considers the gauge transformations of perturbative quantities, choosing a gauge by requiring that certain quantities vanish, rendering all other variables gauge invariant. In the KN2010 formalism, one decomposes the metric tensor into a gauge variant and gauge invariant part from the outset. We compare the two approaches in both the longitudinal and uniform curvature gauges. In the longitudinal gauge, we find that Nakamura's gauge invariant variables correspond exactly to those in the longitudinal gauge (i.e., for scalar perturbations, to the Bardeen potentials), and in the uniform curvature gauge we obtain the usual relationship between gauge invariant variables in the flat and longitudinal gauge. Thus, we show that these two approaches are equivalent.

1. Introduction

Many problems in Physics and Applied Mathematics can be described by non-linear systems of evolution equations. These are notoriously difficult to solve exactly because of the non-linearity. An example of such a theory is General Relativity. Einstein's equations are highly non-linear and can only be solved exactly in a small number of useful cases. To go beyond these solutions perturbative methods are used. Given a solution to the equations in the form of a metric $g^{(0)}_{\mu\nu}$ we assume that we can approximate a neighbouring, more general, solution $g_{\mu\nu}$ using a power series. Thus we express the more general solution in the form

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + \frac{1}{2} g^{(2)}_{\mu\nu} + \dots . \tag{1.1}$$

The metric $g^{(0)}_{\mu\nu}$ is called the *background* and the remaining terms are *perturbations* of the background. The first order part is

$$g_{\mu\nu} - g^{(0)}_{\mu\nu} \simeq g^{(1)}_{\mu\nu} , \tag{1.2}$$

where the remaining terms are assumed to be negligible compared to $g^{(1)}_{\mu\nu}$ and they are neglected at *first order*. In a similar way the higher-order perturbations can be identified, so at second order we have

$$g_{\mu\nu} - g^{(0)}_{\mu\nu} - g^{(1)}_{\mu\nu} \simeq g^{(2)}_{\mu\nu} , \tag{1.3}$$

and so on. This can be described simply if we introduce an infinitesimal parameter $\epsilon \ll 1$ for the perturbation and assume that the series can be written as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon \bar{g}^{(1)}_{\mu\nu} + \frac{1}{2} \epsilon^2 \bar{g}^{(2)}_{\mu\nu} + \dots , \tag{1.4}$$

where the quantities with bars have absolute magnitudes less than one. In this format the orders correspond to the powers of ϵ. In practise it is often a nuisance to introduce the parameter ϵ so we will use the form (1.1) if there is no confusion. Issues of convergence can, in general, be removed by working in a small enough neighbourhood but they should not be ignored.

Having set up the approximation (1.1) we have to substitute it into the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} , \tag{1.5}$$

to obtain solutions of the required order of approximation. This is more complicated than might be expected. Perturbations of the metric imply perturbations of the energy momentum tensor and vice versa but more significantly, calculation of the connection coefficients and the Ricci tensor involve raising and lowering indices and so introduces more terms and potentially couple terms of different orders. At zeroth and first order

this is not a problem but at higher orders it makes calculations much more complicated. Even at second order there are "proper" second order terms, for example $\bar{g}^{(2)}_{\mu\nu}$, and terms quadratic in the first order quantities, for example $\bar{g}^{(1)}_{\mu\nu}$.

In addition to the complexity of the system of equations, the split of the metric and matter variables into a background and perturbations introduces spurious coordinate artefacts or gauge modes as described in detail in numerous reviews [1, 2, 3, 4, 5]. We only give a brief explanation, here.

As we are interested in cosmology, we will assume that the real universe is described by a perturbed Friedmann-Robertson-Walker (FRW) metric and that the background, or unperturbed spacetime, is described by an exact FRW metric. So we have in effect two spacetimes – one "physical" and one "fictitious". We label points in the background by coordinates $\{x^\mu\}$ and a one-to-one-map between points in the background and points in the physical spacetime maps these coordinates from the background to the physical spacetime. We refer to this one-to-one-map as a gauge choice. A change in the map is called a gauge transformation and this may be carried out in a number of ways, see for instance the recent reviews [6, 4, 5]. A perturbation of some quantity is the difference between the value at a point in the physical spacetime and the value at the point in the background with the same coordinates. Clearly such a perturbation depends on the above gauge choice.

It is important to note that a gauge transformation is different to a coordinate transformation which changes the labels on points in the physical and background spacetimes together, and so it does not change the gauge. A simple example of a gauge transformation is an implementation of a coordinate change in the physical spacetime but not in the background. This changes the correspondence between the points in the two spacetimes, so it is a gauge transformation. It is easy to see that a scalar, e.g. the energy density ρ, which is (at least) time dependent, will not be invariant under such a transformation. Furthermore if a gauge is chosen to simplify the metric on the physical spacetime and some residual gauge freedom remains then spurious gauge mode solutions may appear. For these reasons gauge invariant formulations of perturbations and other special gauge fixing forms have been sought. These fall into two broad classes, (1) those following the general pattern of Bardeen [7, 1, 2, 3, 4, 5] and (2) those following the covariant form developed by Ellis and Bruni [8, 9, 10, 11].

At first order in the perturbations the approaches in class (1) above differ largely due to different splits of the spatial part of the metric, and notation and sign conventions. At second order things are more complicated and there are at least two different approaches. Relating the approach used by Malik and Wands [4] and the Nakamura approach [5] (hereafter referred to as MW2009 and KN2010, respectively) is the aim of this paper. We aim at highlighting similarities and differences of the two subclasses of

approaches following Bardeen and try to keep the mathematical background discussions to a minimum, referring the interested reader to the reviews listed above and the original literature cited therein. However, in order to introduce some of the quantities and concepts used later on, we here briefly review the formulation of perturbation theory in a more rigorous sense.

In relativistic perturbation theory, we consider two distinct spacetimes: the 'background' spacetime, denoted $\mathcal{M}_0$ and the 'physical' spacetime denoted $\mathcal{M}$, following Stewart [26]. The physical manifold is nature itself, and we want to describe the properties of this spacetime through the perturbations. On the other hand, the background spacetime is a mere reference spacetime on which to calculate perturbations. We then introduce a point identification map $\mathcal{X} : \mathcal{M}_0 \to \mathcal{M}$ relating points on the unperturbed manifold to points on the perturbed manifold. By virtue of the pullback of this point identification map (denoted $\mathcal{X}^*$), we may treat a tensor field on the physical spacetime $\mathbf{T}$ as a tensor field on the background spacetime $\mathcal{X}^*\mathbf{T}$, which we will often denote, equivalently, as ${}_{\mathcal{X}}\mathbf{T}$. The choice of map $\mathcal{X}$, the *gauge choice*, is not unique; if we choose another different map, $\mathcal{Y}$, the pulled back variables are then different representations of the tensor $\mathbf{T}$. These two gauge choices induce a diffeomorphism $\Phi : \mathcal{M}_0 \to \mathcal{M}_0$, defined as $\Phi := \mathcal{X}^{-1} \circ \mathcal{Y}$.

We conclude this introduction by presenting some essential equations from the reviews [6, 4], in the notation which is used in the standard literature (e.g. Refs. [1, 2, 3]). As in the case of the metric tensor Eq. (1.1) above, we assume that any tensorial quantity $\mathbf{T}$ can be expanded into a power series by

$$\begin{aligned}\mathbf{T} &= \mathbf{T}_0 + \delta\mathbf{T}\,,\\ \delta\mathbf{T} &= \mathbf{T}_1 + \frac{1}{2}\mathbf{T}_2 + \frac{1}{3!}\mathbf{T}_3 + \dots\,,\end{aligned} \tag{1.6}$$

where the subscripts denote the order of the perturbation. The change in a perturbed quantity (to a certain order) induced by a gauge transformation is given by the exponential map, once the generating vector of the gauge transformation, ξ^μ, has been specified. The exponential map is

$$\Phi^*\mathbf{T} \equiv \widetilde{\mathbf{T}} = e^{\pounds_\xi}\mathbf{T}\,, \tag{1.7}$$

where $\pounds_\xi$ denotes the Lie derivative with respect to ξ^μ. Expanding the exponential map and using Eq. (1.6) we obtain in the background, at first order and at second order, respectively,

$$\begin{aligned}\widetilde{\mathbf{T}}_0 &= \mathbf{T}_0\,,\\ \widetilde{\mathbf{T}}_1 &= \mathbf{T}_1 + \pounds_{\xi_1}\mathbf{T}_0\,,\\ \widetilde{\mathbf{T}}_2 &= \mathbf{T}_2 + \pounds_{\xi_2}\mathbf{T}_0 + \pounds^2_{\xi_1}\mathbf{T}_0 + 2\pounds_{\xi_1}\mathbf{T}_1\,,\end{aligned} \tag{1.8}$$

where ξ^μ is the vector field generating the transformation and is expanded order by order as $\xi^\mu \equiv \epsilon\xi_1^\mu + \frac{1}{2}\epsilon^2\xi_2^\mu + O(\epsilon^3)$. The exponential map can also be applied to the coordinates x^μ to obtain the following relationship between coordinates at two points, p and q

$$x^\mu(q) = e^{\xi^\lambda \frac{\partial}{\partial x^\lambda}\big|_p}\, x^\mu(p)\,. \tag{1.9}$$

Expanding to second order gives

$$x^\mu(q) = x^\mu(p) + \epsilon\xi_1^\mu(p) + \frac{1}{2}\epsilon^2\Big(\xi^\mu_{1,\lambda}(p)\xi_1^\lambda(p) + \xi_2^\mu(p)\Big)\,. \tag{1.10}$$

Note that this coordinate relationship is not required to perform calculations in the active approach, but will be useful for later discussion.

On the other hand, when evaluating the gauge transformation rule, a different point of view is adopted in the formulation in KN2010, where the change in a perturbed quantity induced by a gauge transformation Φ is represented in the general form of the Taylor expansion of Φ^*

$$\mathbf{T}(q) = (\Phi^*\mathbf{T})(p) = \mathbf{T}(p) + \epsilon\, \pounds_{\xi_1}\mathbf{T}\big|_p + \frac{1}{2}\epsilon^2\, \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right)\mathbf{T}\Big|_p + O(\epsilon^3). \tag{1.11}$$

As shown by Bruni and coworkers [12, 13, 14, 15, 16], the Taylor expansion of the pull-back of tensor field is always given in the form of Eq. (1.6), even if Φ^* is not an exponential map. In this sense, the Taylor expansion (1.11) represents the perturbative expansion of a wider class of diffeomorphisms than exponential maps. Through this general formula (1.11) and the perturbative expansion (1.6) of the variable $\mathbf{T}$, we can reach the same order-by-order gauge transformation rules as Eq. (1.8).

Although the gauge transformation rules at each order appear to have the same form in the two different formulations [4, 5], we should point out that the approaches to obtain the transformations are conceptually different. As mentioned above, the Taylor expansion (1.11) is valid for a wider class of diffeomorphisms beyond the exponential map. Although we may regard Eq. (1.11) as that of an exponential map through a special choice of ξ_2^μ (e.g. Ref. [12]), there is no guarantee that this is true for any choice of ξ_2^μ. However it seems that, when working to a particular order in perturbation theory, one can always use an exponential map that generates the specified gauge transformation to that order. Therefore, while conceptually the gauge transformation rules used by the different formulations are not the same, this is only a philosophical issue and, in practice, the two are equivalent.

In this paper, we discuss the equivalence of these two formulations in MW2009 and KN2010, through clarifying the correspondence of the variables. Having introduced the basics, we now go on to tackle the main topic of this article, namely, obtaining the relationship between the second-order metric perturbation in MW2009 and that in KN2010. It should be noted that the two formulations use very different notation. We

should note that they use very different notation in these two formulations. Rather than try and enforce a common notation between the two, we present each in its conventional notation, relating them to one another at the end. Further, we discuss two different gauge fixing. One is the Poisson gauge (longitudinal gauge) choice and the other is the flat gauge choice. In MW2009, they showed that these two gauge fixings are complete gauge-fixing, while there is no explicit gauge-fixing in the formulation in KN2010. We clarify the correspondence between the variables in MW2009 and KN2010 through these two gauge-fixing.

We also have to emphasise that although this paper is not a complete survey of the many different formulations of second-order cosmological perturbation theory, the correspondence which is clarified here is a useful check of the equivalence of different formulations and will hopefully lead to a consensus in the community.

This paper is organised as follows. In the next section, we describe first order perturbations, first in the MW2009 approach and then in the KN2010 approach. In §3 we review second order perturbation theory, again starting with the MW2009 approach and then the KN2010 approach. In §4 we compare the two approaches, in both the longitudinal (or Poisson) gauge, and the uniform curvature gauge. Finally, we summarise our results in §5.

If not otherwise stated we use conformal time, η, related to coordinate time t by $dt = ad\eta$, where $a(\eta)$ is the scale factor, see Eq. (2.1), throughout. Derivatives with respect to conformal time are denoted by a prime, and the Hubble parameter is defined, in terms of conformal time, as $H = a'/a$. Greek indices, μ, ν, λ, run from $0 \ldots 3$, and lower case Latin indices, i, j, k, run from $1 \ldots 3$. We also make use of abstract indices a, b, c, in parts.

2. First order cosmological perturbations

The background spacetime $\mathcal{M}_0$ considered in cosmological perturbation theory is a homogeneous, isotropic Friedmann-Robertson-Walker universe foliated by three dimensional hypersurfaces $\Sigma(\eta)$, parametrised by conformal time η. In this paper we restrict ourselves to considering flat spatial hypersurfaces, and the line element for this spacetime is

$$ds^2 = a^2(\eta)\Big[- d\eta^2 + \delta_{ij}dx^i dx^j \Big] \,, \tag{2.1}$$

where $a = a(\eta)$ is the scale factor and δ_{ij} is the metric on the flat space. The full spacetime metric is then expanded, as in Eq. (1.6), as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g^{(1)}_{\mu\nu} + \frac{1}{2}\delta g^{(2)}_{\mu\nu} + \cdots \,,$$

in the notation of MW2009. Alternatively, one can represent the background spacetime metric as

$$g_{ab} = a^2(\eta)\Big[-(d\eta)_a(d\eta)_b + \delta_{ij}(dx^i)_a(dx^j)_b\Big] , \tag{2.2}$$

in the abstract index notation with the full spacetime metric then being expanded as

$$g_{ab} = g^{(0)}_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3). \tag{2.3}$$

Equations (2.2) and (2.3) are the abstract index notation in KN2010.

The metric components at each order in perturbation theory can then be expanded into scalar, vector and tensor components, according to their transformation behaviour on spatial hypersurfaces.

2.1. MW2009 formulation

At linear, or first, order in perturbation theory the general scalar, vector and tensor perturbations to the flat ($K = 0$) FRW background spacetime can be expressed in the line element

$$ds^2 = a^2(\eta)\left[-(1+2\phi_1)d\eta^2 + 2B_{1i}dx^i d\eta + (\delta_{ij} + 2C_{1ij})dx^i dx^j\right] . \tag{2.4}$$

The perturbations of the spatial components of the metric can then be further decomposed as [4]

$$B_{1i} = B_{1,i} - S_{1i}\, , \tag{2.5}$$

$$C_{1ij} = -\psi_1\delta_{ij} + E_{1,ij} + F_{1(i,j)} + \frac{1}{2}h_{1ij}\, , \tag{2.6}$$

where ϕ_1, B_1, ψ_1 and E_1 are scalar metric perturbations, S_{1i} and F_{1i} are divergence-free vector perturbations, and h_{ij} is a transverse, traceless tensor perturbation. In the notation of KN2010, this is then

$$h_{\eta\eta} = -2a^2\phi_1, \tag{2.7}$$

$$h_{i\eta} = a^2 B_{1i} \equiv a^2 D_i B_1 - a^2 S_{1i}, \tag{2.8}$$

$$h_{ij} = 2a^2 C_{1ij} = 2a^2\left(-\psi_1\delta_{ij} + D_iD_jE_1 + D_{(i}F_{1j)} + \frac{1}{2}h_{1ij}\right) , \tag{2.9}$$

where D_i is formally the covariant derivative associated with the spatial metric δ_{ij} and, in practice for this work, it reduces to a partial derivative denoted by a comma. Consistently perturbing the spacetime will naturally invoke perturbations to its matter content as well. In the following, however, we shall only use the energy density.

Before studying the transformation behaviour of perturbations at first order, we split the generating vector ξ_1^μ into a scalar temporal part α_1 and a spatial scalar and divergence-free vector part, respectively β_1 and ${\gamma_1}^i$, as

$$\xi_1^\mu = (\alpha_1, {\beta_{1,}}^i + {\gamma_1}^i)\, . \tag{2.10}$$

We can then consider transformations of different types of perturbation independently, since they decouple at linear order. For example, Eq. (1.8) implies that the energy density perturbation transforms, at first order, as

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\alpha_1 \,, \tag{2.11}$$

where we have used the fact that the Lie derivative, when acting on a scalar function, is just $\pounds_\xi = \xi^\mu(\partial/\partial x^\mu)$. The transformation behaviour of the metric tensor, noting that the Lie derivative for a type (0,2) tensor is given by

$$\pounds_\xi g_{\mu\nu} = g_{\mu\nu,\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda_{,\ \nu} + g_{\lambda\nu}\xi^\lambda_{,\ \mu} \,, \tag{2.12}$$

is

$$\widetilde{\delta g^{(1)}_{\mu\nu}} = \delta g^{(1)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi_1^\lambda + g^{(0)}_{\mu\lambda}\xi^\lambda_{1\ ,\nu} + g^{(0)}_{\lambda\nu}\xi^\lambda_{1\ ,\mu} \,. \tag{2.13}$$

We can obtain the transformation behaviour of each particular metric function by extracting it, in turn, from the above general expression using the method outlined in e.g. MW2009. As mentioned above, we do not focus on details here but instead quote the results. The scalar metric perturbations transform as

$$\widetilde{\phi_1} = \phi_1 + H\alpha_1 + \alpha_1' \,, \tag{2.14}$$

$$\widetilde{\psi_1} = \psi_1 - H\alpha_1 \,, \tag{2.15}$$

$$\widetilde{B_1} = B_1 - \alpha_1 + \beta_1' \,, \tag{2.16}$$

$$\widetilde{E_1} = E_1 + \beta_1 \,, \tag{2.17}$$

the vector metric perturbations as

$$\widetilde{S_1}^i = S_1{}^i - \gamma_1{}^{i\prime} \,, \tag{2.18}$$

$$\widetilde{F_1}^i = F_1{}^i + \gamma_1{}^i \,, \tag{2.19}$$

and the tensor perturbation, h_{1ij}, is gauge invariant. Finally, the scalar shear, which is defined as $\sigma_1 = E_1' - B_1$, transforms as

$$\widetilde{\sigma_1} = \sigma_1 + \alpha_1 \,, \tag{2.20}$$

which will be useful later when we come to define gauges in §4.

2.2. KN2010 formulation

An alternative approach within perturbation theory was presented in KN2010 [5], where the procedure proposed by KN in 2003 [18] is used to construct gauge invariant variables. Evaluating Eq. (1.11) to first-order perturbation, the gauge transformation rule of the first order metric perturbation h_{ab} is given by

$${}_{\mathcal{Y}}h_{ab} - {}_{\mathcal{X}}h_{ab} =: \pounds_{\xi_1} g_{ab}. \tag{2.21}$$

We decompose the linear metric perturbation, h_{ab}, as

$$h_{ab} =: \mathcal{H}_{ab} + \pounds_X g_{ab}, \tag{2.22}$$

where $\mathcal{H}_{ab}$ and $\pounds_X g_{ab}$ are the gauge invariant and variant parts of the first order metric perturbations [18], respectively and X^a is defined below. That is, under a gauge transformation, these are transformed as

$${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi^a_{(1)}. \tag{2.23}$$

We note that the decomposition in Eq. (2.22) is an assumption, however for perturbations to a FRW spacetime it can be shown to be correct [5].

To proceed, we consider the scalar-vector-tensor decomposition of the components $h_{i\eta}$, h_{ij} of h_{ab} as

$$h_{i\eta} = D_i h_{(VL)} + h_{(V)i} \tag{2.24}$$

$$h_{ij} = a^2 \left\{ h_{(L)}\delta_{ij} + \left(D_i D_j - \frac{1}{3}\delta_{ij}\Delta \right) h_{(TL)} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij} \right\} \tag{2.25}$$

where $\Delta := D^i D_i = \delta^{ij} D_i D_j$. Further $h_{(V)i}$, $h_{(TV)j}$, and $h_{(TT)ij}$ satisfy the properties

$$D^i h_{(V)i} = 0, \quad D^i h_{(TV)i} = 0, h_{(T)}{}^i{}_i := \delta^{ij} h_{(T)ij} = 0, \tag{2.26}$$
$$h_{(TT)ij} = h_{(TT)ji}, \quad D^i h_{(TT)ij} = 0.$$

The generator of the gauge transformation, ξ^a, is also decomposed as

$$\xi_a = \xi_\eta (d\eta)_a + \left(D_i \xi_{(L)} + \xi_{(T)i} \right) (dx^i)_a, \quad D^i \xi_{(T)i} = 0. \tag{2.27}$$

Using Eq. (2.21) we can then obtain gauge transformation rules for the components of h_{ab}:

$$\begin{aligned}
{}_{\mathcal{Y}}h_{\eta\eta} - {}_{\mathcal{X}}h_{\eta\eta} &= 2\left(\partial_\eta - H\right)\xi_\eta, && (2.28)\\
{}_{\mathcal{Y}}h_{(VL)} - {}_{\mathcal{X}}h_{(VL)} &= \xi_\eta + \left(\partial_\eta - 2H\right)\xi_{(L)}, && (2.29)\\
{}_{\mathcal{Y}}h_{(V)i} - {}_{\mathcal{X}}h_{(V)i} &= \left(\partial_\eta - 2H\right)\xi_{(T)i}, && (2.30)\\
a^2{}_{\mathcal{Y}}h_{(L)} - a^2{}_{\mathcal{X}}h_{(L)} &= -2H\xi_\eta + \frac{2}{3}\Delta\xi_{(L)}, && (2.31)\\
a^2{}_{\mathcal{Y}}h_{(TL)} - a^2{}_{\mathcal{X}}h_{(TL)} &= 2\xi_{(L)}, && (2.32)\\
a^2{}_{\mathcal{Y}}h_{(TV)i} - a^2{}_{\mathcal{X}}h_{(TV)i} &= \xi_{(T)i}, && (2.33)\\
a^2{}_{\mathcal{Y}}h_{(TT)ij} - a^2{}_{\mathcal{X}}h_{(TT)ij} &= 0. && (2.34)
\end{aligned}$$

We then inspect these gauge transformation rules and define gauge invariant variables. Firstly, Eq. (2.34) shows that the transverse-traceless part $h_{(TT)ij}$ is itself gauge invariant, as expected. We denote this as

$$\overset{(1)}{\chi}_{ij} := h_{(TT)ij}, \tag{2.35}$$

Secondly, the gauge-transformation rules (2.30) and (2.33) give the transverse vector-mode $\overset{(1)}{\nu}_i$ defined as

$$a^2 \overset{(1)}{\nu}_i \ := h_{(V)i} - a^2\partial_\eta h_{(TV)i}. \tag{2.36}$$

In addition to the vector and tensor modes there are two scalar modes in the first order metric perturbation, h_{ab}. To see this, we first consider the gauge transformation rules (2.29) and (2.32). From these transformation rules, the variable X_η defined by

$$X_\eta := h_{(VL)} - \frac{1}{2}a^2\partial_\eta h_{(TL)} \tag{2.37}$$

transforms as

$${}_{\mathcal{Y}}X_\eta - {}_{\mathcal{X}}X_\eta = \xi_\eta. \tag{2.38}$$

Using this definition of X_η, and inspecting the gauge transformation rule (2.28), we can show that the variable $\overset{(1)}{\Phi}$ defined as

$$-2a^2 \overset{(1)}{\Phi} := h_{\eta\eta} - 2\left(\partial_\eta - H\right)X_\eta \tag{2.39}$$

is gauge invariant. Furthermore, from gauge transformation rules (2.31), (2.32), and (2.38), the variable $\overset{(1)}{\Psi}$ defined by

$$-2a^2 \overset{(1)}{\Psi} := a^2\left(h_{(L)} - \frac{1}{3}\Delta h_{(TL)}\right) + 2HX_\eta \tag{2.40}$$

is gauge invariant. The set of variables $\{\overset{(1)}{\Phi}, \overset{(1)}{\Psi}, \overset{(1)}{\nu}_i, \overset{(1)}{\chi}_{ij}\}$ is the complete set of gauge invariant variables.

We can now write the original metric perturbation, h_{ab}, in terms of these gauge invariant variables as

$$h_{\eta\eta} = -2a^2 \overset{(1)}{\Phi} + 2\left(\partial_\eta - H\right)X_\eta, \tag{2.41}$$

$$h_{\eta i} = a^2 \overset{(1)}{\nu}_i + a^2\partial_\eta h_{(TV)i} + D_i h_{(VL)}, \tag{2.42}$$

$$h_{ij} = -2a^2 \overset{(1)}{\Psi} \delta_{ij} + a^2 \overset{(1)}{\chi}_{ij} + a^2 D_i D_j h_{(TL)} - 2H\bar{X}_\eta\delta_{ij} + 2a^2 D_{(i}h_{(TV)j)} \tag{2.43}$$

From the gauge invariance of the variables $\overset{(1)}{\chi}_{ij}$, $\overset{(1)}{\nu}_i$, $\overset{(1)}{\Phi}$, and $\overset{(1)}{\Psi}$, we may read off the gauge invariant part of h_{ab} as

$$\mathcal{H}_{ab} = a^2\left\{-2 \overset{(1)}{\Phi} (d\eta)_a(d\eta)_b + 2 \overset{(1)}{\nu}_i (d\eta)_{(a}(dx^i)_{b)} + \left(-2 \overset{(1)}{\Psi} \delta_{ij} + \overset{(1)}{\chi}_{ij}\right)(dx^i)_a(dx^j)_b\right\}, \tag{2.44}$$

The remaining gauge dependent parts in Eqs. (2.41)–(2.43) should then be given in the form $\pounds_X g_{ab}$ for some vector field X_a. In fact, such a vector field is

$$X_a := X_\eta (d\eta)_a + a^2 \left(h_{(TV)i} + \frac{1}{2} D_i h_{(TL)} \right) (dx^i)_a, \tag{2.45}$$

where X_η is defined in Eq. (2.37). One can also check that this vector field X_a satisfies Eq. (2.23).

Thus, it has been shown that the decomposition (2.22) for the first order metric perturbation is correct for cosmological perturbations. We should note that, in order to accomplish Eq. (2.22), we have assumed the existence of the Green's function Δ^{-1}, and that the perturbative modes which belong to the kernel of the operator Δ have been neglected. However, we ignore both of these issues in this paper, pointing the diligent reader to, for example, Refs. [19, 20] for more information.

Finally, the correspondence between the variables for linear perturbations for the KN2010 approach presented in this section and those for the MW2009 approach presented in § 2.1 are

$$\begin{aligned}
&h_{\eta\eta} \Leftrightarrow -2a^2\phi_1, \quad h_{(VL)} \Leftrightarrow a^2 B_1, \quad h_{(V)i} \Leftrightarrow -a^2 S_{1i}, \\
&h_{(L)} \Leftrightarrow -2\psi_1 + \frac{2}{3}\Delta E_1, \quad h_{(TL)} \Leftrightarrow 2E_1, \quad h_{(TV)i} \Leftrightarrow F_{1i}, \quad h_{(TT)ij} \Leftrightarrow h_{1ij}, \\
&\xi_\eta \Leftrightarrow -a^2\alpha_1, \quad \xi_{(L)} \Leftrightarrow a^2\beta_1, \quad \xi_{(T)i} \Leftrightarrow a^2\gamma_{1i}.
\end{aligned} \tag{2.46}$$

Note that at first order there is no difference between the two formalisms, as can be seen from Eq. (2.46).

3. Second order cosmological perturbations

Having reviewed linear cosmological perturbation theory in both the MW2009 approach and that in KN2010 above, in this section, we review the formalisms at second order.

3.1. MW2009 formulation

In order to consider cosmological perturbation theory to second order we do not truncate the perturbative expansion after the first term in Eq. (1.1). The metric tensor then has components form

$$\delta g^{(2)}_{00} = -2a^2(\eta)\phi_2\,, \tag{3.1}$$

$$\delta g^{(2)}_{0i} = a^2(\eta)(B_{2,i} - S_{2i})\,, \tag{3.2}$$

$$\delta g^{(2)}_{ij} = a^2(\eta)\Big(-2\psi_2\delta_{ij} + 2E_{2,ij} + 2F_{2(i,j)} + h_{2ij} \Big)\,, \tag{3.3}$$

where the quantities here are analogous to their first order counterparts. At second order we split the generating vector ξ^μ_2, as at first order, as

$$\xi^\mu_2 = (\alpha_2, \beta_{2,}{}^i + \gamma_2{}^i)\,. \tag{3.4}$$

Then, using Eq. (1.8), we find that the second order energy density perturbation transforms as

$$\widetilde{\delta\rho_2} = \delta\rho_2 + \rho_0'\alpha_2 + \alpha_1(\rho_0''\alpha_1 + \rho_0'\alpha_1' + 2\delta\rho_1') + (2\delta\rho_1 + \rho_0'\alpha_1)_{,k}(\beta_{1,}{}^{k} + \gamma_1{}^{k}) \,, \tag{3.5}$$

where we note for the first time here that, while at linear order different types of perturbation (scalar, vector and tensor) decouple, this is no longer true at higher order. This is a crucial qualitative difference between first and second order perturbation theory and can lead to the generation of, for example, second order gravitational waves [21] or vector modes and vorticity [22]. At second order the metric tensor transforms as

$$\begin{aligned}
\widetilde{\delta g^{(2)}_{\mu\nu}} = {}& \delta g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi_2^\lambda + g^{(0)}_{\mu\lambda}\xi_2{}^{\lambda}{}_{,\nu} + g^{(0)}_{\lambda\nu}\xi_2{}^{\lambda}{}_{,\mu} + 2\Big[\delta g^{(1)}_{\mu\nu,\lambda}\xi_1^\lambda + \delta g^{(1)}_{\mu\lambda}\xi_1{}^{\lambda}{}_{,\nu} + \delta g^{(1)}_{\lambda\nu}\xi_1{}^{\lambda}{}_{,\mu}\Big] \\
& + g^{(0)}_{\mu\nu,\lambda\alpha}\xi_1^\lambda\xi_1^\alpha + g^{(0)}_{\mu\nu,\lambda}\xi_1{}^{\lambda}{}_{,\alpha}\xi_1^\alpha + 2\Big[g^{(0)}_{\mu\lambda,\alpha}\xi_1^\alpha\xi_1{}^{\lambda}{}_{,\nu} + g^{(0)}_{\lambda\nu,\alpha}\xi_1^\alpha\xi_1{}^{\lambda}{}_{,\mu} + g^{(0)}_{\lambda\alpha}\xi_1{}^{\lambda}{}_{,\mu}\xi_1{}^{\alpha}{}_{,\nu}\Big] \\
& + g^{(0)}_{\mu\lambda}\left(\xi_1{}^{\lambda}{}_{,\nu\alpha}\xi_1^\alpha + \xi_1{}^{\lambda}{}_{,\alpha}\xi^\alpha_{1,\ \nu}\right) + g^{(0)}_{\lambda\nu}\left(\xi_1{}^{\lambda}{}_{,\mu\alpha}\xi_1^\alpha + \xi_1{}^{\lambda}{}_{,\alpha}\xi^\alpha_{1,\ \mu}\right) \,.
\end{aligned} \tag{3.6}$$

From this we can extract, as at first order, the transformation behaviour of the metric perturbations. Again, we refer to MW2009 [4] for the details and only quote the results in this section. One finds that the scalar metric functions transform as

$$\begin{aligned}
\widetilde{\phi_2} = {}& \phi_2 + H\alpha_2 + {\alpha_2}' + \alpha_1\left[{\alpha_1}'' + 5H{\alpha_1}' + \left(H' + 2H^2\right)\alpha_1 + 4H\phi_1 + 2\phi_1'\right] \\
& + 2{\alpha_1}'\left({\alpha_1}' + 2\phi_1\right) + \xi_{1k}\left({\alpha_1}' + H\alpha_1 + 2\phi_1\right)_{,}{}^{k} + \xi_{1k}'\left[\alpha_{1,}{}^{k} - 2B_{1k} - \xi_1^{k\prime}\right] \,,
\end{aligned} \tag{3.7}$$

$$\widetilde{\psi_2} = \psi_2 - H\alpha_2 - \frac{1}{4}\mathbb{X}^k{}_k + \frac{1}{4}\nabla^{-2}\mathbb{X}^{ij}{}_{,ij} \,, \tag{3.8}$$

$$\widetilde{E_2} = E_2 + \beta_2 + \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathbb{X}^{ij}{}_{,ij} - \frac{1}{4}\nabla^{-2}\mathbb{X}^k{}_k \,, \tag{3.9}$$

$$\widetilde{B_2} = B_2 - \alpha_2 + \beta_2' + \nabla^{-2}\mathbb{X}_{\mathrm{B}}{}^{k}{}_{,k} \,, \tag{3.10}$$

where $\mathbb{X}_{ij}$ and $\mathbb{X}_{\mathrm{B}i}$ are defined as

$$\begin{aligned}
\mathbb{X}_{\mathrm{B}i} \equiv {}& 2\Big[\left(2HB_{1i} + B_{1i}'\right)\alpha_1 + B_{1i,k}\xi_1^k - 2\phi_1\alpha_{1,i} + B_{1k}\xi^k_{1,\ i} + B_{1i}\alpha_1' + 2C_{1ik}\xi_1^{k\prime}\Big] \\
& + 4H\alpha_1\left(\xi_{1i}' - \alpha_{1,i}\right) + \alpha_1'\left(\xi_{1i}' - 3\alpha_{1,i}\right) + \alpha_1\left(\xi_{1i}'' - \alpha_{1,i}'\right) \\
& + \xi_1^{k\prime}\left(\xi_{1i,k} + 2\xi_{1k,i}\right) + \xi_1^k\left(\xi_{1i,k}' - \alpha_{1,ik}\right) - \alpha_{1,k}\xi^k_{1,\ i} \,,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\mathbb{X}_{ij} \equiv {}& 2\Big[\left(H^2 + \frac{a''}{a}\right)\alpha_1^2 + H\left(\alpha_1\alpha_1' + \alpha_{1,k}\xi_1{}^k\right)\Big]\delta_{ij} + 2\left(B_{1i}\alpha_{1,j} + B_{1j}\alpha_{1,i}\right) \\
& + 4\Big[\alpha_1\left(C_{1ij}' + 2HC_{1ij}\right) + C_{1ij,k}\xi_1{}^k + C_{1ik}\xi_1{}^{k}{}_{,j} + C_{1kj}\xi_1{}^{k}{}_{,i}\Big] \\
& + 4H\alpha_1\left(\xi_{1i,j} + \xi_{1j,i}\right) - 2\alpha_{1,i}\alpha_{1,j} + 2\xi_{1k,i}\xi_1{}^{k}{}_{,j} + \alpha_1\left(\xi_{1i,j}' + \xi_{1j,i}'\right) \\
& + \xi_{1i,k}\xi_1{}^{k}{}_{,j} + \xi_{1j,k}\xi_1{}^{k}{}_{,i} + \xi_{1i}'\alpha_{1,j} + \xi_{1j}'\alpha_{1,i} + \left(\xi_{1i,jk} + \xi_{1j,ik}\right)\xi_1{}^k \,.
\end{aligned} \tag{3.12}$$

We furthermore find that the second order metric vector perturbations transform as

$$\widetilde{S_{2i}} = S_{2i} - {\gamma_{2i}}' - \mathbb{X}_{\mathrm{B}i} + \nabla^{-2}\mathbb{X}_{\mathrm{B}}{}^{k}{}_{,ki} \,, \tag{3.13}$$

$$\widetilde{F_{2i}} = F_{2i} + \gamma_{2i} + \nabla^{-2}\mathbb{X}_{ik,}{}^{k} - \nabla^{-2}\nabla^{-2}\mathbb{X}^{kl}{}_{,kli} \,, \tag{3.14}$$

and the tensor perturbation as

$$\widetilde{h}_{2ij} = h_{2ij} + \mathbb{X}_{ij} + \frac{1}{2}\left(\nabla^{-2}\mathbb{X}^{kl}{}_{,kl} - \mathbb{X}^{k}{}_{k}\right)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\mathbb{X}^{kl}{}_{,klij} + \frac{1}{2}\nabla^{-2}\mathbb{X}^{k}{}_{k,ij} - \nabla^{-2}\left(\mathbb{X}_{ik,}{}^{k}{}_{j} + \mathbb{X}_{jk,}{}^{k}{}_{i}\right) . \tag{3.15}$$

Note that the second order tensor perturbation, h_{2ij}, changes under a gauge-transformation [6], unlike the first order tensor perturbation, h_{1ij}. However, we show that the expression (3.15) is itself gauge-invariant as discussed at the end of the next section. Hence, using Eq. (3.15), we can construct gauge-invariant tensor perturbations at second order

3.2. KN2010 formulation

We now consider the construction of the gauge-invariant variables for second-order metric perturbation l_{ab} in KN2010 [5]. As we have confirmed the decomposition of the first-order metric perturbation, h_{ab}, into the gauge-invariant $\mathcal{H}_{ab}$ and gauge-variant parts X_a, we can now also find gauge invariant variables for higher-order perturbations [18]. Using Eq. (1.8), the gauge transformation rule for l_{ab} is

$${}_{\mathcal{Y}}l_{ab} - {}_{\mathcal{X}}l_{ab} = +2\pounds_{\xi_1}{}_{\mathcal{X}}h_{ab} + \left(\pounds_{\xi_2} + \pounds_{\xi_1}^2\right)g_{ab}. \tag{3.16}$$

Through the first order gauge-variant variable X_a, we first define the tensor field $\hat{L}_{ab}$ as

$${}_{\mathcal{X}}\hat{L}_{ab} := {}_{\mathcal{X}}l_{ab} - 2\pounds_{{}_{\mathcal{X}}X}{}_{\mathcal{X}}h_{ab} + \pounds_{{}_{\mathcal{X}}X}^2 g_{ab} \tag{3.17}$$

so that it transforms as ${}_{\mathcal{Y}}\hat{L}_{ab} - {}_{\mathcal{X}}\hat{L}_{ab} = \pounds_{\sigma}g_{ab}$, where $\sigma^a := \xi_2^a + [\xi_1, {}_{\mathcal{X}}X]^a$. Since σ^a is an arbitary vector field on $\mathcal{M}_0$, this gauge-transformation rule is the same as that for the first order metric perturbation, h_{ab} (2.21). Using a similar procedure to that with which we decomposed h_{ab}, the variable $\hat{L}_{ab}$ is then $\hat{L}_{ab} =: \mathcal{L}_{ab} + \pounds_Y g_{ab}$. Thus, the decomposition of the second order metric perturbation, l_{ab}, is

$$l_{ab} =: \mathcal{L}_{ab} + 2\pounds_X h_{ab} + \left(\pounds_Y - \pounds_X^2\right)g_{ab}, \tag{3.18}$$

where $\mathcal{L}_{ab}$ and Y^a are the gauge invariant and variant parts of the second order metric perturbations, i.e.,

$${}_{\mathcal{Y}}\mathcal{L}_{ab} - {}_{\mathcal{X}}\mathcal{L}_{ab} = 0, \quad {}_{\mathcal{Y}}Y^a - {}_{\mathcal{X}}Y^a = \xi_{(2)}^a + [\xi_{(1)}, {}_{\mathcal{X}}X]^a. \tag{3.19}$$

We may then choose the components of the gauge invariant variables $\mathcal{L}_{ab}$ in Eq. (3.18) as

$$\mathcal{L}_{ab} = a^2\left\{-2\overset{(2)}{\Phi}(d\eta)_a(d\eta)_b + 2\overset{(2)}{\nu}_i(d\eta)_{(a}(dx^i)_{b)} + \left(-2\overset{(2)}{\Psi}\delta_{ij} + \overset{(2)}{\chi}_{ij}\right)(dx^i)_a(dx^j)_b\right\}, \tag{3.20}$$

where $\overset{(2)}{\nu}_i$ and $\overset{(2)}{\chi}_{ij}$ satisfy the equations

$$D^i \overset{(2)}{\nu}_i = 0, \quad \overset{(2)}{\chi}{}^i{}_i = 0, \quad D^i \overset{(2)}{\chi}_{ij} = 0. \tag{3.21}$$

The gauge invariant variables $\overset{(2)}{\Phi}$ and $\overset{(2)}{\Psi}$ are the second order scalar perturbations and $\overset{(2)}{\nu}_i$ and $\overset{(2)}{\chi}_{ij}$ are the second order vector and tensor modes of the metric perturbations, respectively. Furthermore, using X^a and Y^a, the gauge invariant variables for an arbitrary tensor field Q can then be defined:

$$^{(1)}\mathcal{Q} := {}^{(1)}Q - \pounds_X Q_0, \tag{3.22}$$

$$^{(2)}\mathcal{Q} := {}^{(2)}Q - 2\pounds_X {}^{(1)}Q - \left\{\pounds_Y - \pounds_X^2\right\} Q_0. \tag{3.23}$$

Thus, from Eqs. (3.22) and (3.23), we can see that any variable can be decomposed into a gauge invariant and gauge variant part as

$$^{(1)}Q = {}^{(1)}\mathcal{Q} + \pounds_X Q_0, \tag{3.24}$$

$$^{(2)}Q = {}^{(2)}\mathcal{Q} + 2\pounds_X {}^{(1)}Q + \left\{\pounds_Y - \pounds_X^2\right\} Q_0. \tag{3.25}$$

These decomposition formulae are valid for the perturbations of an arbitrary tensor field without knowing detailed information of the background metric $\overset{(0)}{g}_{ab}$. As a collorary, any equation for the perturbations (e.g. the Einstein equations or the equations of motion for the matter fields) is automatically given in a gauge invariant form [23, 24] due to the lower order equations.

Since we know that any of these equations must be gauge invariant, we only need study the gauge invariant parts of Eq. (3.18) and not treat the components of l_{ab} directly: that is, we must only study $\mathcal{L}_{ab}$. However, the choice of the gauge invariant part at both first order, $\mathcal{H}_{ab}$, and second order, $\mathcal{L}_{ab}$, is not unique. This can be seen as follows: through the components of $\mathcal{H}_{ab}$, we can construct a gauge invariant vector field, for example, $Z_a := -a \overset{(1)}{\Phi} (d\eta)_a + a \overset{(1)}{\nu}_i (dx^i)_a$. Using this we may write

$$h_{ab} = \mathcal{H}_{ab} + \pounds_Z g_{ab} - \pounds_Z g_{ab} + \pounds_X g_{ab} =: \mathcal{K}_{ab} + \pounds_{\tilde{X}} g_{ab}, \tag{3.26}$$

where we have defined $\mathcal{K}_{ab} := \mathcal{H}_{ab} + \pounds_Z g_{ab}$ and $\tilde{X}^a := X^a - Z^a$. Thus there are, in principle, infinitely many choices of $\mathcal{H}_{ab}$, which correspond to the infinitely many choices of the gauge fixing. As mentioned in a previous paper [5], the situation at second order is more complicated; we simply note here that there exist infinitely many choices of $\mathcal{L}_{ab}$. Because of this, it is evident that relating the variables in the MW2009 approach to those in the KN2010 approach is more difficult. However, the procedure is the same as at first order: compare the components of l_{ab} through Eq. (3.18) with Eq. (3.1).

4. Comparison of different formulations

Having now summarised cosmological perturbations from both the MW2009 approach, and that in KN2010, we move to the main point of this paper: linking the two approaches and showing the equivalence between them.

4.1. First order

In this section we present the comparison between the two approaches at linear order. As shown in Eqs. (2.46), at linear order, the correspondence between metric perturbation variables in the MW2009 and KN2010 approach is quite clear. Here we discuss the correspondence of the approaches themselves by choosing two popular gauges: the longitudinal (or Poisson) gauge, and the uniform curvature gauge.

4.1.1. Longitudinal (Poisson) gauge The longitudinal gauge is defined in perturbation theory by choosing hypersurfaces of constant shear ($\widetilde{\sigma}_{1\ell}=0$). Thus, in the longitudinal gauge, the scalar gauge function $\alpha_{1\ell}$ is given by

$$\alpha_{1\ell}=-\sigma_1=B_1-E_1'\,, \tag{4.1}$$

and this gauge is fully specified (for scalars) by requiring separately that $\widetilde{E_{1\ell}}=0$ (which implies that $\widetilde{B_{1\ell}}=0$) and is a natural choice. Hence

$$\beta_{1\ell}=-E_1\,. \tag{4.2}$$

The remaining two scalar metric perturbations, ϕ_1 and ψ_1, are then given as

$$\widetilde{\phi_{1\ell}}=\phi_1+H(B_1-E_1')+(B_1-E_1')'\,, \tag{4.3}$$

$$\widetilde{\psi_{1\ell}}=\psi_1-H\left(B_1-E_1'\right)\,. \tag{4.4}$$

These are the two gauge invariant Bardeen potentials Φ and Ψ [1] and, in fact, these two variables coincide with $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ as defined in KN2010 [5].

By including vector perturbations (an extension of the longitudinal gauge generally called the Poisson gauge) we must also fix the vector gauge function ${\gamma_1}^i$, which can be achieved through the relationship

$$\gamma^i_{1\ell}=\int S^i_1 d\eta+\mathcal{C}^i_1(x^j)\,, \tag{4.5}$$

where $\mathcal{C}^i_1(x^j)$ is an arbitrary constant 3-vector which depends upon the choice of spatial coordinates on an initial hypersurface.

We have thus specified the gauge generating vector, ξ^μ_1 through Eqs. (4.1), (4.2), and (4.5). In this gauge, the remaining components of the linear order metric

perturbation are completely given in the form Eq. (2.44). Therefore, this gauge-fixing corresponds to the choice of the gauge variant part X^a in Eq. (2.22) so that

$$X^a = {}_{\mathcal{P}}X^a = 0, \tag{4.6}$$

where $\mathcal{P}$ denotes the Poisson (longitudinal) gauge choice. In other words, the gauge invariant variables used in the KN2010 approach [5] correspond to the variables associated with the longitudinal gauge.

4.1.2. Uniform curvature (spatially flat) gauge An alternative gauge choice is the uniform curvature, or spatially flat gauge. This amounts to choosing a spatial hypersurface on which the metric is unperturbed by scalar or vector perturbations, which requires $\widetilde{\psi_{1\text{flat}}} = \widetilde{E_{1\text{flat}}} = 0$ and $\widetilde{F^i_{1\text{flat}}} = \mathbf{0}$. This gives the gauge transformation

$$\alpha_{1\text{flat}} = \frac{\psi_1}{H}\,, \quad \beta_{1\text{flat}} = -E_1\,, \quad \gamma^i_{1\text{flat}} = -F^i_1\,. \tag{4.7}$$

The remaining scalar metric perturbations are then

$$\widetilde{\phi_{1\text{flat}}} = \phi_1 + \psi_1 + \left(\frac{\psi_1}{H}\right)'\,, \tag{4.8}$$

$$\widetilde{B_{1\text{flat}}} = B_1 - E_1' - \frac{\psi_1}{H}\,, \tag{4.9}$$

which are gauge invariant. Thus, in the uniform curvature gauge, the metric perturbations as expressed in Eqs. (2.7), (2.8) and (2.9) are

$$\widetilde{h_{\text{flat}\eta\eta}} = -2a^2\widetilde{\phi_{1\text{flat}}}\,, \tag{4.10}$$

$$\widetilde{h_{\text{flat}i\eta}} = 2a^2\left(\widetilde{B_{1\text{flat},i}} - \widetilde{S_{1\text{flat}i}}\right)\,, \tag{4.11}$$

$$\widetilde{h_{\text{flat}ij}} = a^2 h_{1ij}\,. \tag{4.12}$$

Now, we compare this with the KN2010 formalism, in which the gauge choice is regarded as a choice of the gauge variant part $X_a = {}_{\mathcal{F}}X_a$, where $\mathcal{F}$ denotes the flat gauge choice. From Eqs. (2.22), (2.44), and Eqs. (4.10)–(4.12) we obtain

$$-2a^2 \overset{(1)}{\Phi} + 2\partial_\eta({}_{\mathcal{F}}X_\eta) - 2H({}_{\mathcal{F}}X_\eta) = -2a^2\widetilde{\phi_{1\text{flat}}}\,, \tag{4.13}$$

$$a^2 \overset{(1)}{\nu}_i + D_i({}_{\mathcal{F}}X_\eta) + \partial_\eta({}_{\mathcal{F}}X_i) - 2H({}_{\mathcal{F}}X_i) = 2a^2\left(\widetilde{B_{1\text{flat},i}} - \widetilde{S_{1\text{flat}i}}\right), \tag{4.14}$$

$$-2a^2 \overset{(1)}{\Psi} \delta_{ij} + a^2 \overset{(1)}{\chi}_{ij} + D_i({}_{\mathcal{F}}X_j) + D_j({}_{\mathcal{F}}X_i) - 2H\delta_{ij}({}_{\mathcal{F}}X_\eta) = a^2 h_{1ij} \tag{4.15}$$

Since the transverse traceless part of both sides of Eq. (4.15) should coincide with one another, we may identify

$$\overset{(1)}{\chi}_{ij} = h_{1ij}. \tag{4.16}$$

Further, the trace and the longitudinal part of Eq. (4.15) is given by

$$-a^2 \overset{(1)}{\Psi} + \frac{1}{3} D^i({}_{\mathcal{F}}X_i) - H({}_{\mathcal{F}}X_\eta) = 0, \tag{4.17}$$

$$D_i({}_{\mathcal{F}}X_j) + D_j({}_{\mathcal{F}}X_i) = 0\,, \tag{4.18}$$

from which we can choose

$${}_{\mathcal{F}}X_i = 0. \tag{4.19}$$

Further, Eq. (4.17) yields

$${}_{\mathcal{F}}X_\eta = -\frac{a^2}{H} \overset{(1)}{\Psi}\,. \tag{4.20}$$

Then, identifying the divergenceless part of the both sides of Eq. (4.14) with one another gives

$$\widetilde{S_{1\text{flat}\,i}} = -\overset{(1)}{\nu}_i\,. \tag{4.21}$$

Finally, the scalar parts of Eq. (4.14) and Eq. (4.13) yield

$$\widetilde{B_{1\text{flat}}} = \frac{1}{a^2} {}_{\mathcal{F}}X_\eta = -\frac{1}{H} \overset{(1)}{\Psi}\,, \tag{4.22}$$

$$\widetilde{\phi_{1\text{flat}}} = \overset{(1)}{\Phi} + \left(1 - \frac{\partial_\eta H}{H^2}\right) \overset{(1)}{\Psi} + \frac{1}{H}\partial_\eta \overset{(1)}{\Psi}\,. \tag{4.23}$$

As expected, this is simply the relationship between the scalar metric perturbations in the uniform curvature gauge and the Bardeen potentials, the scalar metric perturbations in the longitudinal gauge.

4.2. Second order

4.2.1. Longitudinal (Poisson) gauge In order to extend the Poisson gauge to second order, we continue as at first order. First, we require that $\widetilde{E_{2\ell}} = 0$, which fixes the scalar part of the spatial gauge as

$$\beta_{2\ell} = -E_2 - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathbb{X}^{ij}{}_{,ij} + \frac{1}{4}\nabla^{-2}\mathbb{X}^k{}_k\,. \tag{4.24}$$

Then, requiring that $\widetilde{B_{2\ell}} = 0$ sets $\alpha_{2\ell}$ through Eq. (3.10), and requiring that the vector $\widetilde{F^i_{2\ell}} = \mathbf{0}$ can be used to fix the vector part of the spatial gauge transformation, up to a constant of integration as at linear order. The second order analogues of the gauge invariant Bardeen potentials Φ and Ψ are then given as

$$\begin{aligned}\widetilde{\phi_{2\ell}} &= \phi_2 + H\alpha_{2\ell} + {\alpha_{2\ell}}' + \alpha_{1\ell}\left[{\alpha_{1\ell}}'' + 5H{\alpha_{1\ell}}' + \left(H' + 2H^2\right)\alpha_{1\ell} + 4H\phi_1 + 2\phi_1'\right] \\ &\quad + 2{\alpha_{1\ell}}'\left({\alpha_{1\ell}}' + 2\phi_1\right) + \xi_{1\ell k}\left({\alpha_{1\ell}}' + H\alpha_{1\ell} + 2\phi_1\right)_{,}{}^{k} + \xi'_{1\ell k}\left[\alpha_{1\ell,}{}^{k} - 2B_{1k} - {\xi^k_{1\ell}}'\right]. \\ \widetilde{\psi_{2\ell}} &= \psi_2 - H\alpha_{2\ell} - \frac{1}{4}\mathbb{X}^k_{\ell k} + \frac{1}{4}\nabla^{-2}\mathbb{X}_\ell{}^{ij}{}_{,ij}\,,\end{aligned} \tag{4.25}$$

where $\mathbb{X}_{\ell ij}$ is denotes the quadratic first order terms in Eq. (3.12) using the longitudinal gauge transformation components $\alpha_{1\ell}$ and $\xi^i_{1\ell}$.

To summarise, the second order metric perturbation has the components

$$\tilde{l}_{(lg)\eta\eta} = -2a^2\widetilde{\phi_{2\ell}}, \tag{4.26}$$
$$\tilde{l}_{(lg)i\eta} = -a^2\widetilde{S_{2i}}, \tag{4.27}$$
$$\tilde{l}_{(lg)\eta\eta} = a^2(\eta)\Big(-2\widetilde{\psi_2}\delta_{ij} + \widetilde{h}_{2ij}\Big). \tag{4.28}$$

As at first order, the choice of the Poisson gauge in the MW2009 approach corresponds to the choice

$$Y^a = {}_{\mathcal{P}}Y^a = 0\,. \tag{4.29}$$

in the KN2010 approach. Thus, we again confirmed that the gauge invariant variables used in the KN2010 approach [5] correspond to the variables associated with the longitudinal gauge, with the correspondence between the variables of the MW2009 formulation and KN2010 formulation as

$$\overset{(2)}{\Phi} \Leftrightarrow \widetilde{\phi_{2\ell}}, \quad \overset{(2)}{\nu}_i \Leftrightarrow \widetilde{S_{2i}}, \quad \overset{(2)}{\Psi} \Leftrightarrow \widetilde{\psi_2}, \quad \overset{(2)}{\chi}_{ij} \Leftrightarrow \widetilde{h_{2ij}} \tag{4.30}$$

4.2.2. Uniform curvature (spatially flat) gauge At second order, the gauge condition $\widetilde{\psi_2} = 0$ gives, using Eq. (3.8),

$$\alpha_{2\mathrm{flat}} = \frac{\psi_2}{H} + \frac{1}{4H}\left[\nabla^{-2}\mathbb{X}^{ij}_{\mathrm{flat},ij} - \mathbb{X}^k_{\mathrm{flat}k}\right], \tag{4.31}$$

where we get $\mathbb{X}_{\mathrm{flat}ij}$ from Eq. (3.12) using the first order gauge generators given above, as

$$\begin{aligned}\mathbb{X}_{\mathrm{flat}ij} = {}& 2\left[\psi_1\left(\frac{\psi_1'}{H} + 2\psi_1\right) + \psi_{1,k}\xi^k_{1\mathrm{flat}}\right]\delta_{ij} + \frac{4}{H}\psi_1\left(C_{1ij}' + 2HC_{1ij}\right) \\ & + 4C_{1ij,k}\xi^k_{1\mathrm{flat}} + \left(4C_{1ik} + \xi_{1\mathrm{flat}i,k}\right)\xi^k_{1\mathrm{flat},j} + \left(4C_{1jk} + \xi_{1\mathrm{flat}j,k}\right)\xi^k_{1\mathrm{flat},i} \\ & + \frac{1}{H}\Big[\psi_{1,i}\left(2B_{1j} + \xi_{1\mathrm{flat}j}'\right) + \psi_{1,j}\left(2B_{1i} + \xi_{1\mathrm{flat}i}'\right)\Big] - \frac{2}{H^2}\psi_{1,i}\psi_{1,j} \\ & + \frac{2}{H}\psi_1\left(\xi_{1\mathrm{flat}(i,j)}' + 4H\xi_{1\mathrm{flat}(i,j)}\right) + 2\xi^k_{1\mathrm{flat}}\xi_{1\mathrm{flat}(i,j)k} + 2\xi_{1\mathrm{flat}k,i}\xi^k_{1\mathrm{flat},j}\,,\end{aligned} \tag{4.32}$$

where we define

$$\xi_{1\mathrm{flat}i} = -\left(E_{1,i} + F_{1i}\right). \tag{4.33}$$

Finally, the gauge conditions $\widetilde{E_{2\mathrm{flat}}} = 0$ and $\widetilde{F^i_{2\mathrm{flat}}} = \mathbf{0}$ enable us to specify the gauge functions $\beta_{2\mathrm{flat}}$ and $\gamma^i_{2\mathrm{flat}}$ completely. Thus, in the uniform curvature gauge, the metric

perturbation has the components

$$\tilde{l}_{(\mathrm{flat})\eta\eta} = -2a^2\widetilde{\phi_{2\mathrm{flat}}}, \tag{4.34}$$

$$\tilde{l}_{(\mathrm{flat})\eta i} = a^2(\eta)\left(\widetilde{B_{2\mathrm{flat},i}} - \widetilde{S_{2\mathrm{flat}i}}\right), \tag{4.35}$$

$$\tilde{l}_{(\mathrm{flat})ij} = a^2(\eta)\widetilde{h_{2\mathrm{flat}ij}}, \tag{4.36}$$

On the other hand, according to the decomposition (3.18), the second-order metric perturbation l_{ab} in the flat gauge is given by

$${}_{\mathcal{F}}l_{ab} =: \mathcal{L}_{ab} + 2\pounds_{{}_{\mathcal{F}}X}\,{}_{\mathcal{F}}h_{ab} + \left(\pounds_{{}_{\mathcal{F}}Y} - \pounds^2_{{}_{\mathcal{F}}X}\right)g_{ab}. \tag{4.37}$$

Here, the components of ${}_{\mathcal{F}}X_a$ are given by Eqs. (4.19) and (4.20), and the components of ${}_{\mathcal{F}}h_{ab}$ are given by Eqs. (4.10)–(4.12) with the relations (4.16), (4.21), (4.22), and (4.23). Tedious calculations show that the components of the terms

$${}_{\mathcal{F}}\Xi_{ab} := 2\pounds_{{}_{\mathcal{F}}X}\,{}_{\mathcal{F}}h_{ab} - \pounds^2_{{}_{\mathcal{F}}X}g_{ab} \tag{4.38}$$

in Eq. (4.37) are

$$\begin{aligned}
{}_{\mathcal{F}}\Xi_{\eta\eta} := \frac{2a^2}{H^4}\Bigg(&-4H^4 \stackrel{(1)}{\Phi}\stackrel{(1)}{\Psi} -2H^4(\stackrel{(1)}{\Psi})^2 - 2H^2(\partial_\eta \stackrel{(1)}{\Psi})^2 - 2H^3 \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\Phi} \\
&-4H^3 \stackrel{(1)}{\Phi} \partial_\eta \stackrel{(1)}{\Psi} -5H^3 \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\Psi} +H\partial_\eta^2 H(\stackrel{(1)}{\Psi})^2 - 4(\partial_\eta H)^2(\stackrel{(1)}{\Psi})^2 \\
&+4H^2\partial_\eta H \stackrel{(1)}{\Phi}\stackrel{(1)}{\Psi} +4H^2\partial_\eta H(\stackrel{(1)}{\Psi})^2 + 6H\partial_\eta H \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\Psi} \\
&-H^2 \stackrel{(1)}{\Psi} \partial_\eta^2 \stackrel{(1)}{\Psi}\Bigg),
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
{}_{\mathcal{F}}\Xi_{i\eta} := \frac{a^2}{H^3}\Bigg[&8\partial_\eta H \stackrel{(1)}{\Psi} D_i \stackrel{(1)}{\Psi} -3H \stackrel{(1)}{\Psi} \partial_\eta D_i \stackrel{(1)}{\Psi} -5H\partial_\eta \stackrel{(1)}{\Psi} D_i \stackrel{(1)}{\Psi} \\
&-4H^2 \stackrel{(1)}{\Phi} D_i \stackrel{(1)}{\Psi} -8H^2 \stackrel{(1)}{\Psi} D_i \stackrel{(1)}{\Psi} \\
&+8H^3 \stackrel{(1)}{\Psi}\stackrel{(1)}{\nu}_i +4H^2 \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\nu}_i -4H\partial_\eta H \stackrel{(1)}{\Psi}\stackrel{(1)}{\nu}_i \\
&+4H^2\partial_\eta \stackrel{(1)}{\Psi}\stackrel{(1)}{\nu}_i\Bigg],
\end{aligned} \tag{4.40}$$

$$\begin{aligned}
{}_{\mathcal{F}}\Xi_{ij} := \frac{2a^2}{H^2}\Bigg(&-3D_i \stackrel{(1)}{\Psi} D_j \stackrel{(1)}{\Psi} -2H^2\gamma_{ij}\left(\stackrel{(1)}{\Psi}\right)^2 - H\gamma_{ij} \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\Psi} \\
&+2HD_i \stackrel{(1)}{\Psi}\stackrel{(1)}{\nu}_j +2HD_j \stackrel{(1)}{\Psi}\stackrel{(1)}{\nu}_i \\
&+H \stackrel{(1)}{\Psi} \partial_\eta \stackrel{(1)}{\chi}_{ij} +2H^2 \stackrel{(1)}{\Psi}\stackrel{(1)}{\chi}_{ij}\Bigg).
\end{aligned} \tag{4.41}$$

Together with Eqs. (4.34)–(4.36), we obtain the components of Eq. (4.37) as follows:

$$-2a^2\widetilde{\phi_{2\mathrm{flat}}} = -2a^2 \overset{(2)}{\Phi} + 2\partial_\eta {}_{\mathcal{F}}Y_\eta - 2H {}_{\mathcal{F}}Y_\eta + {}_{\mathcal{F}}\Xi_{\eta\eta}, \tag{4.42}$$

$$a^2(\eta)\left(\widetilde{B_{2\mathrm{flat},i}} - \widetilde{S_{2\mathrm{flat}i}}\right) = a^2 \overset{(2)}{\nu}_i + \partial_\eta {}_{\mathcal{F}}Y_i + D_i {}_{\mathcal{F}}Y_\eta - 2H {}_{\mathcal{F}}Y_i + {}_{\mathcal{F}}\Xi_{i\eta}, \tag{4.43}$$

$$\begin{aligned} a^2(\eta)\widetilde{h_{2\mathrm{flat}ij}} &= -2a^2 \overset{(2)}{\Psi}\gamma_{ij} + a^2 \overset{(2)}{\chi}_{ij} \\ &\quad + D_i {}_{\mathcal{F}}Y_j + D_j {}_{\mathcal{F}}Y_i - 2H\gamma_{ij} {}_{\mathcal{F}}Y_\eta + {}_{\mathcal{F}}\Xi_{ij}. \end{aligned} \tag{4.44}$$

The trace part of Eq. (4.44) is given by

$$0 = -6a^2 \overset{(2)}{\Psi} + 2D^k {}_{\mathcal{F}}Y_k - 6H {}_{\mathcal{F}}Y_\eta + \gamma^{ij} {}_{\mathcal{F}}\Xi_{ij} \tag{4.45}$$

and the traceless part is

$$\begin{aligned} a^2(\eta)\widetilde{h_{2\mathrm{flat}ij}} &= a^2 \overset{(2)}{\chi}_{ij} + D_i {}_{\mathcal{F}}Y_j + D_j {}_{\mathcal{F}}Y_i - \frac{2}{3}\gamma_{ij} D^k {}_{\mathcal{F}}Y_k \\ &\quad + {}_{\mathcal{F}}\Xi_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl} {}_{\mathcal{F}}\Xi_{kl}, \end{aligned} \tag{4.46}$$

where ${}_{\mathcal{F}}Y_j$ is decomposed as

$${}_{\mathcal{F}}Y_j =: D_j {}_{\mathcal{F}}Y_{(L)} + {}_{\mathcal{F}}Y_{(V)j}, \quad D^j {}_{\mathcal{F}}Y_{(V)j} = 0. \tag{4.47}$$

Eq. (4.46) is then

$$\begin{aligned} a^2(\eta)\widetilde{h_{2\mathrm{flat}ij}} &= +a^2 \overset{(2)}{\chi}_{ij} + 2\left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) {}_{\mathcal{F}}Y_{(L)} + 2D_{(i} {}_{\mathcal{F}}Y_{(V)j)} \\ &\quad + {}_{\mathcal{F}}\Xi_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl} {}_{\mathcal{F}}\Xi_{kl}, \end{aligned} \tag{4.48}$$

where we have used the fact that D_i is the covariant derivative associated with the flat metric. Taking the divergence of Eq. (4.48), we obtain

$$\frac{4}{3}D_j \Delta {}_{\mathcal{F}}Y_{(L)} + \Delta {}_{\mathcal{F}}Y_{(V)j} + D^i\left({}_{\mathcal{F}}\Xi_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl} {}_{\mathcal{F}}\Xi_{kl}\right) = 0\,, \tag{4.49}$$

and further taking the divergence of Eq. (4.49), gives

$${}_{\mathcal{F}}Y_{(L)} = -\frac{3}{4}\Delta^{-2} D^j D^i \left({}_{\mathcal{F}}\Xi_{ij} - \frac{1}{3}\gamma_{ij}\gamma^{kl} {}_{\mathcal{F}}\Xi_{kl}\right). \tag{4.50}$$

Substituting Eq. (4.50) into Eq. (4.49), we obtain

$${}_{\mathcal{F}}Y_{(V)j} = \Delta^{-1}\left[D_j \Delta^{-1} D^k D^l - \gamma_j{}^l D^k\right]\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn} {}_{\mathcal{F}}\Xi_{mn}\right]. \tag{4.51}$$

and hence we have

$${}_{\mathcal{F}}Y_j = \Delta^{-1}\left[\frac{1}{4}D_j \Delta^{-1} D^k D^l - \gamma_j{}^l D^k\right]\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn} {}_{\mathcal{F}}\Xi_{mn}\right]. \tag{4.52}$$

Since

$$D^k {}_{\mathcal{F}}Y_k = -\frac{3}{4}\Delta^{-1}D^lD^k\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right], \tag{4.53}$$

Eq. (4.46) yields

$$\begin{aligned}\widetilde{h_{2\text{flat}ij}} = &\stackrel{(2)}{\chi}_{ij} \\ &+ \frac{1}{2a^2}\left[\Delta^{-1}\left(D_iD_j\Delta^{-1}D^kD^l - 4\gamma_{(i}{}^{l}D_{j)}D^k + \gamma_{ij}D^lD^k\right) + 2\gamma_i{}^k\gamma_j{}^l\right] \\ &\quad\times\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right].\end{aligned} \tag{4.54}$$

On the other hand, the trace part of Eq. (4.45) gives

$${}_{\mathcal{F}}Y_\eta = -\frac{a^2}{H}\stackrel{(2)}{\Psi} - \frac{1}{4H}\Delta^{-1}D^lD^k\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right] + \frac{1}{6H}\gamma^{ij}{}_{\mathcal{F}}\Xi_{ij} \tag{4.55}$$

Taking the divergence of Eq. (4.43) and substituting Eqs. (4.52), (4.53), and (4.55), we obtain

$$\begin{aligned}\widetilde{B_{2\text{flat}}} = &-\frac{1}{H}\stackrel{(2)}{\Psi} - \frac{1}{4a^2}\left[3\left(\partial_\eta - 2H\right)\Delta^{-1} + \frac{1}{H}\right]\Delta^{-1}D^lD^k\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right] \\ &+ \frac{1}{6a^2H}\gamma^{ij}{}_{\mathcal{F}}\Xi_{ij} + \frac{1}{a^2}\Delta^{-1}D^i{}_{\mathcal{F}}\Xi_{i\eta}.\end{aligned} \tag{4.56}$$

and substituting Eq. (4.56) into Eq. (4.43) gives

$$\begin{aligned}\widetilde{S_{2\text{flat}i}} = &-\stackrel{(2)}{\nu}_i \\ &-\frac{1}{a^2}\left(\partial_\eta - 2H\right)\Delta^{-1}\left[D_i\Delta^{-1}D^lD^k - \gamma_i{}^lD^k\right]\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right] \\ &-\frac{1}{a^2}\left[{}_{\mathcal{F}}\Xi_{i\eta} - D_i\Delta^{-1}D^k{}_{\mathcal{F}}\Xi_{k\eta}\right].\end{aligned} \tag{4.57}$$

Finally, through Eq. (4.42), we obtain

$$\begin{aligned}\widetilde{\phi_{2\text{flat}}} = &\stackrel{(2)}{\Phi} + \frac{1}{a^2}\left(\partial_\eta - H\right)\left(\frac{a^2}{H}\stackrel{(2)}{\Psi}\right) \\ &-\frac{1}{a^2}\left(\partial_\eta - H\right)\left(-\frac{1}{4H}\Delta^{-1}D^lD^k\left[{}_{\mathcal{F}}\Xi_{kl} - \frac{1}{3}\gamma_{kl}\gamma^{mn}{}_{\mathcal{F}}\Xi_{mn}\right] + \frac{1}{6H}\gamma^{ij}{}_{\mathcal{F}}\Xi_{ij}\right) \\ &-\frac{1}{2a^2}{}_{\mathcal{F}}\Xi_{\eta\eta}.\end{aligned} \tag{4.58}$$

As at linear order, Eqs. (4.54), (4.56), (4.57), and (4.58) give the relationship between between the variables in uniform curvature gauge and the Poisson gauge.

Before concluding this section, we note the non-uniqueness of the gauge invariant variables defined in Eqs. (2.22) and (3.18) as discussed at the end of §3.2. In this section, we regard the choice of gauge as the specification of the vector fields X^a and Y^a, which are the gauge-variant part of the first and second order metric perturbations,

respectively. On the other hand, we may also regard gauge-fixing as the specification of the gauge invariant parts $\mathcal{H}_{ab}$ and $\mathcal{L}_{ab}$ of the metric perturbations in the sense discussed at the end of §3.2. Actually, we may use ${}_{\mathcal{F}}X_a$, whose components are given by Eqs (4.19) and (4.20), as the vector field Z_a in Eq. (3.26), since the components of ${}_{\mathcal{F}}X_a$ are specified by the gauge invariant variables. In this case, the first order metric perturbation h_{ab} is decomposed into gauge invariant part $\mathcal{K}_{ab}$ and gauge-variant part $\tilde{X}^a$ as Eq. (3.26), where $\mathcal{K}_{ab}$ and $\tilde{X}^a$ are given by

$$\mathcal{K}_{ab} = \mathcal{H}_{ab} + \pounds_{{}_{\mathcal{F}}X}\, g_{ab}, \quad \tilde{X}^a = X^a - {}_{\mathcal{F}}X^a. \tag{4.59}$$

Further components of $\mathcal{K}_{ab} = {}_{\mathcal{F}}h_{ab}$ are given by Eqs. (4.10)–(4.12). $\mathcal{K}_{ab}$ is regarded as the realisation of the gauge invariant variables for the first-order metric perturbation associated with the flat gauge.

Furthermore, in the case of the second-order metric perturbation l_{ab}, we define

$$\mathcal{J}_{ab} := \mathcal{L}_{ab} + 2\pounds_{{}_{\mathcal{F}}X}\, {}_{\mathcal{F}}h_{ab} + \left(\pounds_{{}_{\mathcal{F}}Y} - \pounds^2_{{}_{\mathcal{F}}X}\right) g_{ab} \tag{4.60}$$

and Eq. (4.37) is given by

$$l_{ab} = \mathcal{J}_{ab} + 2\pounds_{\tilde{X}} h_{ab} + \left(\pounds_{Y - {}_{\mathcal{F}}Y + [X, \tilde{X}]} - \pounds^2_{\tilde{X}}\right) g_{ab}. \tag{4.61}$$

Then, choosing

$$\tilde{Y}^a = Y^a - {}_{\mathcal{F}}Y^a + [X, \tilde{X}]^a, \tag{4.62}$$

the second-order metric perturbation l_{ab} is given by

$$l_{ab} = \mathcal{J}_{ab} + 2\pounds_{\tilde{X}} h_{ab} + \left(\pounds_{\tilde{Y}} - \pounds^2_{\tilde{X}}\right) g_{ab}. \tag{4.63}$$

The tensor $\mathcal{J}_{ab}$ is clearly gauge invariant and $\tilde{Y}^a$ satisfy the gauge transformation rule

$${}_{\mathcal{Y}}\tilde{Y}^a - {}_{\mathcal{X}}\tilde{Y}^a = \xi^a_{(2)} + [\xi_{(1)}, {}_{\mathcal{X}}\tilde{X}]^a \tag{4.64}$$

under the transformation $\Phi = \mathcal{X}^{-1} \circ \mathcal{Y}$, i.e., $\tilde{Y}^a$ satisfy the property (3.19) of the gauge-variant part of the second order metric perturbation. The components of the gauge invariant part $\mathcal{J}_{ab}$ are given by Eqs. (4.34)–(4.36) and so $\mathcal{J}_{ab}$ is regarded as the realisation of the gauge invariant variables for the second order metric perturbation associated with the flat gauge.

At the end of §3.1 we noted that the second order tensor perturbation is gauge dependent. Therefore, the expression for the gauge invariant second order tensor perturbation will differ depending on the choice of gauge. This is due to the fact that in perturbation theory beyond linear order, in which mode-couplings occur due to the non-linearity of the system, the notion of the transverse-traceless part of the metric perturbation cannot be identified uniquely with the gravitational waves. As shown for the second order case in Section 3.1, the gauge invariant higher order transverse-traceless perturbation has contributions from the second order tensor perturbation and

other first order metric potentials. In other words, we might say that the second order gravitational waves are also generated from the the first order gravitational potentials as discussed in literature [21].

5. Summary and discussions

In this paper, we have compared and contrasted two different approaches to metric based cosmological perturbation theory, and have derived the relationship between the standard approach á la Bardeen [1], which was discussed in MW2009 [4], and the formalism studied in KN2010 [5]. We started by introducing the basics of relativistic perturbation theory, and then presented first- and second-order cosmological perturbation theory in the MW2009 and KN2010 approach, respectively. Finally, in §4, we compared the two formalisms directly for both the longitudinal or Poisson gauge and the uniform curvature gauge.

Both approaches recognise the spurious gauge artefacts present in perturbation theory, and adopt different techniques in order to remove them. In the MW2009 approach, one perturbs the metric tensor and then inspects the gauge transformations of the metric perturbations, using these to eliminate the gauge freedom and hence construct gauge invariant variables. The KN2010 approach splits the perturbation to the metric into a gauge invariant and gauge variant part, and then writes all equations in terms of these gauge invariant variables. In the MW2009 approach, the gauge choice is made by specifying the gauge generating vector, ξ^μ. On the other hand, in the KN2010 approach the gauge is defined through the gauge variant vector X^a (or, at second order, Y^a). We showed that the gauge invariant variables that are used in KN2010 are equivalent to the usual gauge invariant variables of the Poisson gauge (so that the Poisson gauge is specified through $X^a = 0 = Y^a$). When relating KN2010 formalism to the MW2009 approach in the uniform curvature gauge, we simply obtain the usual relationship between the gauge invariant variables in the Poisson gauge and those in the uniform curvature gauge. Thus, we have shown that the two approaches are equivalent.

While this result is not necessarily surprising, since both the approaches are based upon metric cosmological perturbation theory, showing this equivalence is a good consistency check for the KN2010 approach. Furthermore, there may be certain problems for which it could be advantageous to use one approach over the other. Having shown this equivalence, and having a working knowledge of both theories, may enable one to more easily solve the problem in hand.

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